THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050B Mathematical Analysis I
Tutorial 7

Date: 24 October, 2024

1. (Exercise 3.5.2 of [BS11]) Show directly from the definition that the following are Cauchy sequences

   (a) $\left(\dfrac{n+1}{n}\right)$

   (b) $\left(\dfrac{1}{1!}+\dfrac{1}{2!}+\cdots+\dfrac{1}{n!}\right)$

2. (Exercise 3.5.9 of [BS11]) If $0<r<1$ and $|x_{n+1}-x_n|<r^n$ for all $n\in\mathbb{N}$, show that (x_n) is a Cauchy sequence.

3. Use either the $\varepsilon-\delta$ definition of limit or the Sequential Criterion for limits, to establish the following:

   (a) $\lim\limits_{x\to 0}\dfrac{x^2}{|x|}=0$

   (b) $\lim\limits_{x\to 1}\dfrac{x^2-x+1}{x+1}=\dfrac{1}{2}$

   (c) $\lim\limits_{x\to 0}\dfrac{1}{\sqrt{x}}$, $(x>0)$ does not exist.

4. (Exercise 4.3.8 of [BS11]) Let f be defined on $(0,\infty)$ to $\mathbb{R}$. Prove that $\lim\limits_{x\to\infty}f(x)=L$ if and only if $\lim\limits_{x\to 0^+}f(1/x)=L$.

Announcement: Midterm 31/10 2:30pm – 4:15pm.

Begin discussion at 2:55pm.

1. (Exercise 3.5.2 of [BS11]) Show directly from the definition that the following are Cauchy sequences

   (a) $\left(\dfrac{n+1}{n}\right)$

   (b) $\left(\dfrac{1}{1!}+\dfrac{1}{2!}+\cdots+\dfrac{1}{n!}\right)$

Pf: a) Let $\varepsilon>0$. Then notice that $\frac{n+1}{n}=1+\frac{1}{n}$.

Then WLOG assume $m>n$, then we have

$$\left|1+\frac{1}{m}-1-\frac{1}{n}\right|=\left|\frac{1}{m}-\frac{1}{n}\right|\leq\frac{1}{m}+\frac{1}{n}\underset{\substack{\uparrow\\ m>n}}{\leq}\frac{2}{n}.$$

So taking $N>\frac{2}{\varepsilon}$ works.

b) Let $\varepsilon>0$ be given. Again, assume $m>n$.

Note that for $k\geq 4$, $2^k<k!$

Then

$$\left|\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{m!}-\left(\frac{1}{1!}+\frac{1}{2!}+\cdots\frac{1}{n!}\right)\right|$$

$$=\left|\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\cdots+\frac{1}{m!}\right|$$

$$\leq\frac{1}{2^{n+1}}+\cdots+\frac{1}{2^m}$$

$$\leq\frac{1}{2^n}<\frac{1}{n}.$$ Then taking $N>\frac{1}{\varepsilon}$ works.

2. (Exercise 3.5.9 of [BS11]) If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is a Cauchy sequence.

Pf: Suffices to show $|x_m - x_n| \to 0$ as $n \to \infty$. with $m > n$.

So
$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \cdots + x_{n+1} - x_n|$$
$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$
$$\leq r^m + r^{m-1} + r^{m-2} + \cdots + r^n$$
geometric sequence $= \dfrac{r^n}{1-r} \to 0$ as $n \to \infty$ b/c. $0 < r < 1$.

3. Use either the $\varepsilon - \delta$ definition of limit or the Sequential Criterion for limits, to establish the following:

   (a) $\lim_{x \to 0} \frac{x^2}{|x|} = 0$

   (b) $\lim_{x \to 1} \frac{x^2 - x + 1}{x + 1} = \frac{1}{2}$

   (c) $\lim_{x \to 0} \frac{1}{\sqrt{x}}$, $(x > 0)$ does not exist.

Pf: a) let $\varepsilon > 0$ be given.

$$\left| \frac{x^2}{|x|} - 0 \right| = \left| \frac{x^2}{|x|} \right| = |x|.$$ So taking $\delta = \varepsilon$, we have for all $0 < |x - 0| < \delta = \varepsilon$,

$$\left| \frac{x^2}{|x|} - 0 \right| = |x| < \delta = \varepsilon.$$

b) let $\varepsilon > 0$ be given.

$$\left| \frac{x^2 - x + 1}{x + 1} - \frac{1}{2} \right| = \left| \frac{(2x-1)(x-1)}{2(x+1)} \right|$$

$$\leq |x - 1| \left| \frac{2x - 1}{2x + 2} \right|$$

Since $x \to 1$, can eventually take $0 < \frac{1}{2} < x$.

$$\leq |x - 1| \left| \frac{\cancel{2x}}{\cancel{2x}} \right|$$

So taking $\delta = \varepsilon$, we have for $0 < |x - 1| < \delta$,

$$\left| \frac{x^2 - x + 1}{x + 1} - 1 \right| < \varepsilon.$$

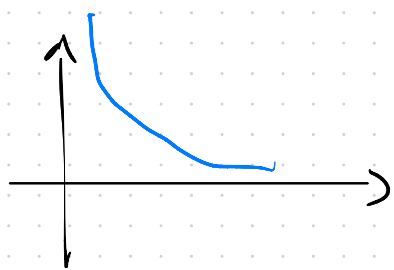
c) Show $\lim_{x \to 0} \frac{1}{\sqrt{x}}$ $(x > 0)$ DNE.

By sequential criterion, suffices to show $\exists\ (x_n)$ s.t. $x_n > 0$ and $x_n \to 0$ s.t.

$\frac{1}{\sqrt{x_n}}$ Does not converge.

Take $x_n = \frac{1}{n^2}$. Then $x_n > 0$ and $x_n \to 0$ as $n \to \infty$.

and $\frac{1}{\sqrt{x_n}} = \frac{1}{\sqrt{\frac{1}{n^2}}} = n$ which clearly diverges to $+\infty$.



4. (Exercise 4.3.8 of [BS11]) Let f be defined on $(0, \infty)$ to $\mathbb{R}$. Prove that $\lim_{x\to\infty} f(x) = L$ if and only if $\lim_{x\to 0^+} f(1/x) = L$.

Pf: First sps $\lim_{x\to+\infty} f(x) = L$. Let $\varepsilon > 0$ be given. There is a $K(\varepsilon) > 0$ s.t. for any $x > K$,

$$|f(x) - L| < \varepsilon.$$

So taking $\delta = \frac{1}{K}$, when $x > K > 0$, $0 < \left|\frac{1}{x}\right| < \delta$, and so

$$\left|f\left(\tfrac{1}{x}\right) - L\right| < \varepsilon. \quad \text{So } f\left(\tfrac{1}{x}\right) \to L \text{ as } x \to 0^+.$$

Now sps $\lim_{x\to 0^+} f\left(\frac{1}{x}\right) = L$. Then $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. whenever $0 < \left|\frac{1}{x}\right| < \delta$, $\left|f\left(\frac{1}{x}\right) - L\right| < \varepsilon$.

Then taking $K = \frac{1}{\delta}$, for $0 < x < \delta$, we have $0 < K < \frac{1}{x} =: z$

$$|f(z) - L| = \left|f\left(\tfrac{1}{x}\right) - L\right| < \varepsilon.$$